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ORIGINAL PAPER

When is a cellular-countably-compact space, countably compact?

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Abstract
We continue the study of cellular-compact spaces and the larger class of cellular-countably-compact spaces. We give a number of sufficient conditions involving local bases and local π-bases in order that a cellular-countably-compact space be countably compact and some conditions which imply that a topology is maximal with respect to being cellular-countably-compact are obtained. We also consider the compact productivity of the previously mentioned properties and give a characterization of those spaces whose product with a compact space is almost cellular-countably-compact.

Keywords Countably compact space · Feebly compact space · Cellular-countably-compact space · Almost cellular-countably-compact space · Countable closed-pseudocharacter · G_δ-diagonal · Maximal cellular-countably-compact space · Compact productivity

Mathematics Subject Classification Primary 54D99; Secondary 54D20

1 Introduction, notation and terminology

Throughout the paper, all spaces are assumed to be Hausdorff and whenever a stronger separation axiom is needed, it will be specified. If P is a topological property, then a space X is said to be *cellular-P* (respectively, *almost cellular-P*) if whenever $\mathcal{U}$ is a family of mutually disjoint non-empty open sets (such a family will be called a *cellular family of open sets*), there is a subspace with property P which meets every element (respectively $|\mathcal{U}|$-many elements) of the family $\mathcal{U}$.

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The class of cellular-Lindelöf spaces, which was first introduced by Bella and Spadaro in [8], contains both the class of Lindelöf spaces and that of *ccc* spaces. and has recently been studied by a number of authors, for example see [9, 11, 23–25].

More generally, the cardinal function cellularity, $c(X)$, and cellular families have frequently been employed to obtain interesting cardinal inequalities; we mention some recent articles on this theme. In [7], a generalization of the well-known inequality $|X| \leq 2^{c(X)\chi(X)}$ was given and related inequalities concerning spaces with a π-base whose elements have compact closure were recently obtained in [6]. In [9], it was shown that a monotonically normal cellular-Lindelöf space is Lindelöf, and a cardinality bound for cellular-Lindelöf spaces with a regular G_δ-diagonal was given. In [19], the class of star-cellular-Lindelöf spaces is studied and it was shown that every first countable, star-cellular-Lindelöf, perfect T_2-space has cardinality at most $\mathfrak{c}$, giving a partial answer to Question 4 of [8].

Recently, interest in cellular-compactness has arisen; this property was first introduced in [21] and further studied in [2]; the broader class of almost cellular-compact spaces was introduced in [4]. An Isbell-Mrowka Ψ-space is easily seen to be almost cellular-compact but not cellular-compact, nor cellular-countably-compact. This latter property was first introduced and studied in [2] where among other results it was shown that a Urysohn, first countable, cellular-countably-compact space is countably compact and the question was asked whether this result extends to the class of all first countable spaces. In Sect. 2, of this paper we give a strong positive answer to this question and also show that a Tychonoff cellular-countably-compact space with a G_δ-diagonal is compact and metrizable; this result corrects an error in the proof of Proposition 5.16 of [2].

In Sect. 3 we study maximality of the property of being cellular-countably-compact. This topic was briefly studied in [2] where it was shown that a sequential, cellular-countably-compact space with a disjoint local π-base at each point is maximal cellular-countably-compact. Among other results in this section, we show that the existence of a disjoint local π-base at each point characterizes maximality of this property in the class of regular sequential spaces. As a corollary, it follows that consistently, every compact, sequential space is maximal cellular-countably-compact.

In Sect. 4, we consider the problem of preservation of the properties of being almost cellular-compact and almost cellular-countably-compact under products, another topic very briefly touched on in [2]. In that paper, it was shown in Theorem 5.4 that neither the property of being cellular-compact nor that of being cellular-countably-compact are preserved, in general, under products with compact spaces. However, it is not known whether such products must at least be almost cellular-compact or almost cellular-countably-compact, respectively. In a series of results in this section, we give a number of sufficient conditions for the product of an almost cellular-compact space and a compact space to be almost cellular-compact, but the general problem remains open.

If D is a discrete subspace of a space X, then an *open expansion* of D is a cellular family $\mathcal{U} = \{U_d : d \in D\}$ of open sets such that $d \in U_d$ for each $d \in D$. A discrete subspace D of a topological space X is *strongly discrete* if it has an open expansion. An easy to prove folklore result states that in a T_3-space every infinite subspace contains an infinite strongly discrete subset, but this is not necessarily true in a Hausdorff space. The notation we use and almost all undefined terms are taken from [12], but the definitions of a local π-base and the cardinal functions *cellularity*, $c(X)$, and *extent*, $e(X)$, can be found in [15].

2 When is a cellular-countably-compact space, countably compact?

Definition 2.1 Following [16] we say that a subset A of a topological space X is *fluffy* if there is a cellular family of open sets $\{U_a : a \in A\}$ such that $a \in \mathrm{cl}(U_a)$ for each $a \in A$.

It was shown in [16] that in a Hausdorff space, each infinite subset contains an infinite fluffy subset. Question 5.13 of [2] asked whether a first countable cellular-countably-compact space must be countably compact. We answer this question affirmatively in a strong way in the next theorem which has a conclusion similar to that of Theorem 2.7 of [16], but whose hypothesis is strictly weaker, in the class of all T_3-spaces, than that used in [16]. The result is also a considerable strengthening of Corollary 4.2 of [21].

A space has *countable closed-pseudocharacter* if each of its points is the intersection of a countable subfamily of its closed neighbourhoods. A point p with this property was called an E_1-point in [5] and later in [18].

Theorem 2.2 *If X is a cellular-countably-compact space with countable closed-pseudocharacter, then X is regular, first countable and countably compact.*

Proof Suppose first that $x \in X$ is a non-isolated point and U is an open set such that $x \in \mathrm{cl}(U)$. Let $\{V_n : n \in \omega\}$ be a nested family of open neighbourhoods of x such that $\bigcap\{\mathrm{cl}(V_n) : n \in \omega\} = \{x\}$. For each $n \in \omega$, there is some $m_n > n$ such that $(V_n \cap U)\backslash(\mathrm{cl}(V_{m_n}) \cap U) \neq \emptyset$ for otherwise, for some $k \in \omega$ and all $m > k$, $\mathrm{cl}(V_m) \cap U \supseteq V_k \cap U$ and then,

$$\{x\} = \bigcap\{\mathrm{cl}(V_m) : m > k\} \supseteq \bigcap\{\mathrm{cl}(V_m) \cap U : m > k\} \supseteq V_k \cap U$$

which would imply that x is an isolated point of $\mathrm{cl}(U)$ and hence an isolated point of X. Let $W_0 = (V_0 \cap U)\backslash(\mathrm{cl}(V_{m_0}) \cap U)$ and for each $n \in \omega$, we define recursively

$$W_{n+1} = (V_{m_n} \cap U)\backslash(\mathrm{cl}(V_{m_{n+1}}) \cap U).$$

The sets $\{W_n : n \in \omega\}$ are disjoint non-empty open sets contained in U and $W_{n+1} \subseteq V_{m_n} \subseteq V_n$. Furthermore, since X is cellular-countably-compact, there is some countably compact subspace $C \subseteq X$ such that $C \cap W_n \neq \emptyset$ for each $n \in \omega$. Thus for each $n \in \omega$, we may pick $c_n \in C \cap W_n$ and let $D = \{c_n : n \in \omega\}$. The set D is discrete and since $\mathrm{cl}_C(D)$ is countably compact, D must have an accumulation point $p \in X$. Moreover, if $p \neq x$, then there is some $\ell \in \omega$ such that $p \notin \mathrm{cl}(V_{m_\ell})$ and since all but finitely many of the sets W_n are contained in V_ℓ, this contradicts the fact that p is an accumulation point of D. Thus we have shown that any countably compact subspace of X which meets each element of the family $\{W_n : n \in \omega\}$ must contain the point x.

We now show that every infinite discrete subset $A = \{a_n : n \in \omega\} \subseteq X$ has an accumulation point. There is some infinite fluffy subset $E \subseteq A$ and mutually disjoint open sets $\{U_e : e \in E\}$ such that $e \in \mathrm{cl}(U_e)$ for each $e \in E$; without loss of generality, we assume that if e is isolated, then $U_e = \{e\}$. As in the previous paragraph, for each $e \in E$ we may find a family $\mathcal{W}_e$ of mutually disjoint open sets such that $\bigcup \mathcal{W}_e \subseteq U_e$. Then $\mathfrak{W} = \bigcup\{\mathcal{W}_e : e \in E\}$ is a family of mutually disjoint open sets and hence there is some countably compact subspace $Y \subseteq X$ which meets each element of $\mathfrak{W}$ and hence contains E. Thus E, and hence A, has an accumulation point.

To show that X is a T_3-space suppose that there is some point $p \in X$ and an open neighbourhood V of p which contains no closed neighbourhood of p. Let $\{W_n : n \in \omega\}$ be a countable nested family of open sets such that $\bigcap\{\mathrm{cl}(W_n) : n \in \omega\} = \{p\}$. Clearly $\mathrm{cl}(W_n)\backslash V$ is infinite for each $n \in \omega$ and so we may pick distinct points $x_n \in \mathrm{cl}(W_n)\backslash V$. A straightforward argument now shows that the infinite set $\{x_n : n \in \omega\}$ has no accumulation point, contradicting the fact that X is countably compact.

Finally, using an argument similar to that of the previous paragraph one can show that a feebly compact regular space with countable pseudocharacter is first countable. A proof of this fact can be found in the proof of (b) in Theorem 4 of [13]. □

We note that in the statement of the previous theorem it is not possible to substitute an almost cellular-countable-compact space for a cellular-countably-compact space; as mentioned in Sect. 1, an Isbell-Mrowka Ψ-space is easily seen to be almost cellular-countably-compact but not cellular-countably-compact (and hence not countably compact).

The previous theorem should be compared to Theorem 2.7 of [16], where a different hypothesis (that the closure of every countable strongly discrete subspace is countably compact) is employed to obtain the same conclusion. As a consequence we have the following corollary:

Corollary 2.3 *If X is a space with countable closed-pseudocharacter then the following are equivalent:*

(1) *X is regular, first countable and countably compact;*
(2) *The closure of every strongly discrete countable subset of X is countably compact, and*
(3) *X is cellular-countably-compact.*

The next corollary is an immediate consequence of Theorem 2.2 and Theorem 4.13 of [21], where it was shown that a first countable, cellular-compact, regular space has cardinality at most $\mathfrak{c}$. This result was first proved in [16] and should be compared with that of Theorem 2.10 below.

Corollary 2.4 *If X is a cellular-compact space with countable closed-pseudocharacter, then $|X| \leq \mathfrak{c}$.*

The conditions (2) and (3) in Corollary 2.3 are not equivalent, even in the class of regular radial spaces as we illustrate below. However, if X is an infinite Urysohn space then (as shown in [14]) every infinite subset of X contains an infinite strongly discrete subspace and hence condition (2) in the previous corollary immediately implies that X is countably compact - that is to say (2) $\Rightarrow$ (3) in the class of Urysohn (and hence in the class of regular) spaces. However, a Fréchet, cellular-countably-compact Tychonoff space need not be countably compact and hence need not satisfy (2). In the next example, we recall the properties of a space which appears in a slightly different context in Example 3.24 of [21].

Example 2.5 Let Σ denote the Σ-product in $\{0, 1\}^{\omega_1}$ whose base point is $\overline{0}$, the function which is identically 0. It is well known and easy to see that Σ is countably compact and it follows from Theorem 2.1 of [17] that Σ is Fréchet. Let $X = \Sigma \backslash \{\overline{0}\}$; clearly X is not countably compact, but we will show that X is cellular-compact, hence cellular-countably-compact.

It follows from Theorem 3.13 of [21], that it is sufficient to show that Σ has no disjoint local π-base at $\overline{0}$. Since Σ is dense in $\{0, 1\}^{\omega_1}$ it follows that $c(\Sigma) = \omega$, and so to prove our claim, we need only show that there is no *countable* disjoint local π-base in Σ at the point $\overline{0}$. To this end, suppose that $\mathcal{U} = \{U_n : n \in \omega\}$ is a cellular family of basic open sets in Σ, say $U_n = \bigcap\{\pi_\alpha^{-1}[a_{n\alpha}] : \alpha \in I_n\} \cap \Sigma$ where $a_{n\alpha} \in \{0, 1\}$ and $I_n \subseteq \omega_1$ is finite for each $n \in \omega$; thus $I = \bigcup\{I_n : n \in \omega\}$ is countable and if $\gamma \in \omega_1 \backslash I$, then the open neighbourhood $\pi_\gamma^{-1}[0] \cap \Sigma$ of $\overline{0} \in \Sigma$, contains no element of the family $\mathcal{U}$. □

In case the space has a disjoint local π-base at each point, we are able to prove a positive result. Recall from [14] that a space X is *strongly Hausdorff* if each infinite subset of X

contains an infinite strongly discrete subset; a Urysohn space (and hence a regular space) is strongly Hausdorff. A space X is *Whyburn* (respectively, *weakly Whyburn*) if whenever $A \subseteq X$ and $x \in \mathrm{cl}(A)\backslash A$, there is $B \subseteq A$ such that $\mathrm{cl}(B)\backslash A = \{x\}$ (respectively, whenever A is not closed, there is $B \subseteq A$ such that $|\mathrm{cl}(B)\backslash A| = 1$).

Theorem 2.6 *A strongly Hausdorff, cellular-countably-compact, Whyburn space X which has a disjoint local π-base at each point is countably compact and Fréchet.*

Proof Suppose that $D = \{d_n : n \in \omega\}$ is a countable discrete subset of X; it suffices to show that D is not closed and since X is strongly Hausdorff, we may assume that D is strongly discrete. Thus we may find an open expansion $\{U_n : n \in \omega\}$ of D and for each $n \in \omega$, a disjoint local π-base $\mathcal{P}_n = \{V_{\alpha n} : \alpha \in I_n\}$ at d_n. If d_n is an isolated point of X then we assume that $U_n = \{d_n\}$ and $\mathcal{P}_n = \{\{d_n\}\}$ and without loss of generality, we may assume that $\bigcup \mathcal{P}_n \subseteq U_n$ for each $n \in \omega$. The family of open sets $\mathcal{C} = \bigcup\{\mathcal{P}_n : n \in \omega\}$ is cellular and so there is a countably compact subspace $C \subseteq X$ which meets each element of $\mathcal{C}$, that is to say C meets each set $V_{\alpha n}$ for all $n \in \omega$ and $\alpha \in I_n$. We may then choose $x_{\alpha n} \in V_{\alpha n} \cap C$ for all $n \in \omega$ and all $\alpha \in I_n$; for each $n \in \omega$, the set $\{x_{\alpha n} : \alpha \in I_n\}$ is discrete and since $\mathcal{P}_n$ is a local π-base at d_n, it follows that $d_n \in \mathrm{cl}(\{x_{\alpha n} : \alpha \in I_n\})$ for each $n \in \omega$. Since X is Whyburn, there is a subset $A_n \subseteq \{x_{\alpha n} : \alpha \in I_n\} \subseteq C$ such that $\mathrm{cl}(A_n)\backslash\{x_{\alpha n} : \alpha \in I_n\} = \{d_n\}$ and so d_n is the only accumulation point of A_n, implying that $d_n \in C$ for all $n \in \omega$. Since C is countably compact, it follows immediately that D is not closed. That X is also Fréchet, now follows from Theorem 2.2 of [22] where it was proved that a Whyburn countably compact space is Fréchet. □

Corollary 2.7 *A strongly Hausdorff, cellular-countably-compact, Whyburn space with a dense set of isolated points is Fréchet and countably compact.*

The conditions imposed in Theorem 2.6 seem rather strong, but the following examples illustrate the difficulties involved in trying to weaken them.

Example 2.8 The Tychonoff Plank $T = ((\omega_1 + 1) \times (\omega + 1))\backslash\{(\omega_1, \omega)\}$ is a weakly Whyburn space which has a disjoint local π-base at each point; T is cellular-countably-compact since it has a dense countably compact subspace, but T is not countably compact. □

Example 2.9 Let Z be the countably compact subspace of $\beta\omega$ of cardinality $\mathfrak{c}$ constructed in Example 3.10.19 of [12], $\omega \subseteq Z \subseteq \beta\omega$; we will show that there exists an infinite discrete subset $D \subseteq \beta\omega\backslash Z$ which has no accumulation point in Z. It will then follow that as a subspace of $\beta\omega$, $X = Z \cup D$ is a cellular-countably-compact Tychonoff space with a countable, disjoint, local π-base at each point, which is not countably compact.

Let f be a (necessarily) continuous surjection from ω onto a countable dense subspace of $\{0, 1\}^{\mathfrak{c}}$ and let g be its continuous and (necessarily) surjective extension to $\beta\omega$.

Note first that each point of $\{0, 1\}^{\mathfrak{c}}$ is the limit of an injective sequence in $\{0, 1\}^{\mathfrak{c}}$. Since $|g[Z]| \leq \mathfrak{c} < |\{0, 1\}^{\mathfrak{c}}|$, we may choose a point $g(p) \in \{0, 1\}^{\mathfrak{c}}\backslash g[Z]$ and an injective sequence $\langle g(x_n)\rangle$ in $\{0, 1\}^{\mathfrak{c}}\backslash\{g(p)\}$ which converges to $g(p)$, (where $\{x_n : n \in \omega\} \cup \{p\} \subseteq \beta\omega$). Let $D = \{x_n : n \in \omega\}$; we claim that D has no accumulation point in Z and hence that $X = Z \cup D$ is not countably compact. For if $q \in \beta\omega$ were an accumulation point of D, then since $\langle g(x_n)\rangle \to g(p)$ it would follow that $g(q) = g(p)$, showing that $q \notin Z \cup D$. □

A space X is said to have a *G_δ-diagonal* (respectively, a *regular G_δ-diagonal*) if the diagonal $\Delta = \{(x, x) : x \in X\} \subseteq X \times X$ is the intersection of countably many of its neighbourhoods (respectively, closed neighbourhoods); a space with a regular G_δ-diagonal clearly has countable closed-pseudocharacter. A statement very similar to that of the next theorem first appeared in [2], but the proof given there contained and employed the false statement that a pseudocompact space with a G_δ-diagonal is compact and metrizable. The statement of the next result should be compared to that of Corollary 2.4.

Theorem 2.10 *A cellular-countably-compact space X with a regular G_δ-diagonal is compact and metrizable, and hence has cardinality at most $\mathfrak{c}$.*

Proof Since the space X has a regular G_δ-diagonal, it follows that X has countable closed-pseudocharacter and then by Theorem 2.2, X is a first countable, countably compact, T_3-space which has a G_δ-diagonal. A result of J. Chaber (see [10]) now implies that X is compact and metrizable. □

Question 2.11 *Are the conditions* (2) *and* (3) *of Corollary* 2.3 *equivalent in the class of all spaces with countable pseudocharacter (respectively, in the class of all spaces with a G_δ-diagonal)?*

Definition 2.12 The diagonal Δ of a space X is *small* if for any set $A \subseteq (X \times X)\setminus\Delta$ of cardinality ω_1 there exists a set $B \subseteq A$ such that $|B| = \omega_1$ and $\mathrm{cl}(B) \cap \Delta = \emptyset$.

It is well-known and not hard to see that a G_δ-diagonal is small. The following questions then arise:

Question 2.13 *Is a cellular-countably-compact (T_3-) space with a G_δ-diagonal (respectively, a small diagonal), countably compact?*

However, an almost cellular-countably-compact Tychonoff space with a G_δ-diagonal need not even be cellular-countably-compact; the requisite example is again an Isbell-Mrowka Ψ-space which, being strongly σ-discrete (that is to say, the space is the union of countably-many closed discrete subspaces), has a G_δ-diagonal.

3 Maximal cellular-countably-compact spaces

We say that a space (X, τ) is *maximal cellular-countably-compact* (respectively, *maximal almost cellular-countably-compact*) if it is cellular-countably-compact, but whenever $\sigma \supsetneq \tau$, then (X, σ) is not cellular-countably-compact (respectively, *not almost cellular-countably-compact*). It is far from clear that a cellular-countably-compact topology on a set X can be enlarged to a maximal cellular-countably-compact topology, the space $\beta\omega$ being a case in point as we will show later. However, by imposing conditions on a cellular-countably-compact space X similar to those of Theorems 2.2 and 2.6, maximal cellular-countably-compact topologies can be shown to exist. This topic was mentioned briefly in [2] and previously, several characterizations of maximal feebly compact spaces were obtained in [18] where however, all spaces were only assumed to be T_1 and not necessarily Hausdorff. One such characterization given in Proposition 2.7 of [18] used a property similar to, but strictly weaker than countable closed-pseudocharacter.

Theorem 3.1 *If (X, τ) is a cellular-countably-compact Hausdorff space with countable closed-pseudocharacter, then (X, τ) is maximal cellular-countably-compact.*

Proof It follows from Theorem 2.2 that (X, τ) is a first countable countably compact T_3-space. Then Corollary 2 of [5] implies that such a space is maximal countably compact.

Since any topology σ on X stronger than τ also has countable closed-pseudocharacter, then by the same argument, it follows that if (X, σ) is cellular-countably-compact, it is countably compact. As a consequence, $\sigma = \tau$, which shows that (X, τ) is maximal cellular-countably-compact. □

The next result should be compared with Theorem 5.17 of [2], where it was shown that a sequential cellular-countably-compact space with a disjoint local π-base at each point is maximal cellular-countably-compact.

Theorem 3.2 *If (X, τ) is a Whyburn, strongly Hausdorff, cellular-countably-compact space with a disjoint local π-base at each point, then (X, τ) is maximal cellular-countably-compact.*

Proof It follows from Theorem 2.6 that (X, τ) is countably compact and Fréchet. Suppose that σ is a topology on X which is strictly stronger than τ. Since (X, τ) is Fréchet, there is some injective sequence $\langle x_n \rangle$ which converges in X to a point p in (X, τ) which is not in the range S of $\langle x_n \rangle$, and which has no accumulation point in (X, σ), and hence, S is an infinite, closed and discrete subset of the space (X, σ). Let ξ be the topology on X generated by the sub-base $\tau \cup \{X \backslash S\}$; note that if $p \notin U \in \xi$, then $U \in \tau$. It is clear that (X, ξ) is not countably compact and we proceed to show that it is Whyburn. Since τ and ξ differ only at p, to show that (X, ξ) is Whyburn we need only show that if $p \in \mathrm{cl}_\xi(A) \backslash A$, then there is some $B \subset A$ such that $\mathrm{cl}_\xi(B) \backslash A = \{p\}$. However, if $p \in \mathrm{cl}_\xi(A)$, then since $p \notin \mathrm{cl}_\xi(A \cap S)$ it follows that $p \in \mathrm{cl}_\xi(A \backslash S)$. But then, since τ and ξ coincide on $X \backslash S$ and (X, τ) is Whyburn, it follows that there is some $B \subseteq A \backslash S$ such that $\mathrm{cl}_\xi(B) \backslash A = \mathrm{cl}_\tau(B) \backslash A = \{p\}$.

To show that (X, σ) is not cellular-countably-compact, it clearly suffices to show that (X, ξ) is not cellular-countably-compact. Thus we assume to the contrary, that (X, ξ) is cellular-countably-compact and proceed to show that this produces a contradiction. Since $S \cup \{p\}$ is a compact subspace of (X, τ), a standard argument shows that S has a τ-open expansion $\mathcal{W}$ such that $p \notin \bigcup \mathcal{W}$. Thus $\mathcal{W} = \{U_n : n \in \omega\}$ is a cellular family of ξ-open sets such that $x_n \in U_n$ and $p \notin U_n$ for each $n \in \omega$. In the space (X, τ) we may find a disjoint local π-base $\mathcal{V}_n$ at each of the points x_n in (X, τ) such that $x_n \notin \bigcup \mathcal{V}_n$ and $\bigcup \mathcal{V}_n \subseteq U_n$ for each $n \in \omega$. Since S is ξ-closed, it follows immediately that $\bigcup\{\mathcal{V}_n : n \in \omega\}$ is a disjoint local π-base at p in the space (X, ξ). Thus the space (X, ξ) satisfies the hypotheses of Theorem 2.6, and so by that theorem, (X, ξ) would be countably compact, a contradiction □

The conditions imposed in the two previous theorems might appear strong, but it seems they cannot be relaxed significantly: It is easy to show that none of the spaces exhibited in Examples 2.5, 2.8 and 2.9 are maximal cellular-countably-compact. However, there are maximal cellular-countably-compact spaces with disjoint local π-bases at each point which are not Whyburn and do not have countable closed-pseudocharacter. The one-point compactification of an Isbell-Mrowka space is a sequential, compact space with a dense set of isolated points which is easily seen to be both maximal countably compact and maximal cellular-countably-compact. However, in the class of regular spaces, the condition of having a disjoint local π-base at each point is necessary in order that a space be maximal cellular-countably-compact. We need the following lemma whose proof, which we omit, is almost identical to that of Proposition 3.11 of [21].

Lemma 3.3 *If X is a regular cellular-countably-compact space and there exists $p \in X$ such that there is no disjoint local π-base at p, then $X \backslash \{p\}$ is cellular-countably-compact*

Theorem 3.4 *If (X, τ) is a regular, maximal cellular-countably-compact space then X has a disjoint local π-base at each point.*

Proof If there is some point p which does not have a disjoint local π-base, then p is not an isolated point of X and it follows from the previous lemma that the space (X, σ), where σ is the topology generated by the subbase $\tau \cup \{\{p\}\}$, is cellular-countably-compact. □

Corollary 3.5 *A sequential, cellular-countably-compact, regular space is maximal cellular-countably-compact if and only if it has a disjoint local π-base at each point.*

Proof The sufficiency is Theorem 5.17 of [2] while the necessity follows from the previous theorem. □

Note that the one-point compactification of an Isbell-Mrowka Ψ-space shows that a scattered maximal cellular-countably-compact space need not be Fréchet.

Corollary 3.6 *($2^\omega < 2^{\omega_1}$) Each sequential, compact space is maximal cellular-countably-compact.*

Proof This follows immediately from Corollary 3.23 of [21] and the previous corollary. □

The next theorem is analogous to Lemma 3.26 of [20], but the proof differs somewhat. The result will allow us to show that many cellular-countably-compact topologies cannot be refined to a maximal cellular-countably-compact topology.

Theorem 3.7 *If (X, τ) is a cellular-countably-compact space and there exists a maximal cellular-countably-compact topology $\sigma \supseteq \tau$, then each point of X which is not the limit of an injective sequence in (X, τ), is an isolated point of (X, σ).*

Proof Suppose that $p \in X$ is not an isolated point of (X, σ); it follows that the space $(X \setminus \{p\}, \sigma)$ is not cellular-countably-compact, for if it were, then the topological union of $(X \setminus \{p\}, \sigma)$ with the one-point discrete space $\{p\}$, would produce a cellular-countably-compact topology ξ on X strictly stronger than σ. Thus there is a cellular family $\mathcal{V}$ of non-empty open subsets of $(X \setminus \{p\}, \sigma)$ with the property that no countably compact subspace of $(X \setminus \{p\}, \sigma)$ meets each element of $\mathcal{V}$. However, since each element of $\mathcal{V}$ is open in (X, σ), there is a countably compact subspace C of (X, σ) which meets each element of $\mathcal{V}$; it follows that $p \in C$ and $C \setminus \{p\}$ is not countably compact. Thus there is some countably infinite subset S of $C \setminus \{p\}$ whose only accumulation point is p. Clearly then, S is a sequence which converges to p in (X, σ) and hence also in (X, τ). □

Corollary 3.8 *There is no maximal cellular-countably-compact topology which refines that of the Tychonoff Plank or that of $\beta\omega$.*

Problem 3.9 *Characterize those spaces which are maximal cellular-countably-compact.*

The last theorem of this section shows that even first countability is not sufficient in general, to imply that a topology is maximal almost cellular-countably-compact.

Theorem 3.10 *If (X, τ) is an almost cellular-countably-compact, first countable space without isolated points and $c(X) = \omega$, then (X, τ) is not maximal almost cellular-countably-compact.*

Proof Suppose that σ is a topology on X and $\sigma \supsetneq \tau$. There is some $p \in X$ and an injective sequence $S = \{x_n : n \in \omega\} \subseteq X \setminus \{p\}$ which converges to p in (X, τ) but which is closed and discrete in (X, σ) and then as in Theorem 3.2, denote by ξ the topology on X generated by the sub-base $\tau \cup \{X \setminus S\}$. To show that (X, τ) is not maximal almost cellular-countably-compact, we will show that (X, ξ) is almost cellular-countably-compact. To this end, suppose that that $\mathcal{U} = \{U_n : n \in \omega\}$ is a cellular family of non-empty open sets in (X, ξ). Without loss of generality we may assume that for each $n \in \omega$, $p \notin U_n$.

If p is not an accumulation point of $\mathcal{U}$ in (X, ξ), then there is an open ξ-neighbourhood W of p such that for all but finitely-many $n \in \omega$, $U_n \subseteq X \backslash W$. Then $V = \mathrm{int}_\tau(\mathrm{cl}_\tau(W))$ is an open τ-neighbourhood of p which meets only finitely-many elements of $\mathcal{U}$. There is then a countably compact subspace C of (X, τ) which meets infinitely-many of those elements of $\mathcal{U}$ disjoint from V and since τ and ξ coincide on $X \backslash V$, it follows that $C \backslash V$ is a countably compact subspace of (X, ξ) which meets infinitely many elements of the cellular family $\mathcal{U}$.

If, on the other hand, p is an accumulation point of $\mathcal{U}$, then let $\{V_n : n \in \omega\}$ be a nested local base at p in (X, τ). For each $n \in \omega$, we may find $m_n \in \omega$ such that $U_{m_n} \cap V_n \neq \emptyset$, where we assume that $m_{n+1} > m_n$. Since S is nowhere dense in (X, ξ), we may choose $y_n \in (U_{m_n} \backslash S) \cap V_n$. The subspace $\{y_n : n \in \omega\} \cup \{p\}$ of (X, ξ) is compact and has non-empty intersection with infinitely many elements of $\mathcal{U}$. □

Corollary 3.11 *The Euclidean topology on* $[0, 1]$ *is not maximal almost cellular-countably-compact.*

We do not know if there exists a maximal almost cellular-countably-compact topology on $[0, 1]$ which refines the Euclidean topology.

4 Almost cellular-compactness of products

A space X is said to be *linearly H-closed* if every ascending open cover of X has a dense element. Such a space is necessarily feebly compact, but need not be countably compact - again an Isbell-Mrowka Ψ-space illustrates this fact. In Theorem 2.11 of [3], it was shown that a space is linearly H-closed if and only if every cellular family of regular cardinality of non-empty open sets has a complete accumulation point. It is then natural to ask whether or not in a linearly H-closed space, every infinite cellular family of non-empty open sets has a complete accumulation point. The next result is a partial answer to this question in the class of countably compact spaces.

Theorem 4.1 *If X is a countably compact, linearly H-closed space, then every cellular family of size less than $\aleph_{\omega_1}$ of non-empty open sets has a complete accumulation point.*

Proof Suppose that $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ is a cellular family of non-empty open sets where $\kappa < \aleph_{\omega_1}$. If κ is a regular cardinal, then since X is linearly H-closed, it follows immediately that $\mathcal{U}$ has a complete accumulation point. If on the other hand κ is singular, then $cof(\kappa) = \omega$ and hence we may find a countable set of regular cardinals $\{\lambda_n : n \in \omega\}$ whose supremum is κ. For each $n \in \omega$, let $\mathcal{V}_0 = \{U_\alpha : \alpha < \lambda_0\}$ and $\mathcal{V}_{n+1} = \{U_\alpha : \lambda_n \leq \alpha < \lambda_{n+1}\}$. Since for each $n \in \omega$, $\mathcal{V}_n$ is a cellular family of regular cardinality of non-empty open sets, it follows that each such family $\mathcal{V}_n$ has a complete accumulation point $p_n \in X$. Let $P = \{p_n : n \in \omega\}$; there are now two cases to consider.
(1) If P is infinite, then since X is countably compact, P has an accumulation point $q \in X$ and hence every neighbourhood W of q is such that W contains infinitely many points p_n, say $\{p_{n_k} : k \in \omega\} \subseteq W$ and hence meets λ_{n_k}-many elements of $\mathcal{V}$ for infinitely many $k \in \omega$. The result now follows from the fact that $\sum_{k \in \omega} \lambda_{n_k} = \kappa$.
(2) If P is finite, say $P = \{p_1, \ldots, p_n\}$, then there is some $j \in \omega$ such that p_j is a complete accumulation point of the family $\mathcal{U}_n$ for infinitely many $n \in \omega$. An argument similar to that of case (1) now shows that p_j is a complete accumulation point of the family $\mathcal{U}$. □

Corollary 4.2 *If X is a locally compact, countably compact space and $c(X) < \aleph_{\omega_1}$, then X is linearly H-closed if and only if X is almost cellular-compact.*

Proof Using the notation of the previous theorem, if K is a compact neighbourhood of the complete accumulation point of the cellular family of non-empty open sets $\mathcal{U}$, then K meets $|\mathcal{U}|$-many elements of $\mathcal{U}$. □

We omit the proof of the following theorem which is almost identical to a combination of the proofs of Proposition 5.14 and Theorem 5.15 of [4].

Theorem 4.3 *Let Q be a topological property (the relevant one here being countable compactness); the following are equivalent for each Hausdorff space X:*

(1) *For each family $\mathcal{U} = \{U_\alpha : \alpha \in \kappa\}$ of non-empty open subsets of X, there is a subspace $A \subseteq X$ with property Q which meets κ-many elements of $\mathcal{U}$.*
(2) *For each cardinal κ and every κ-sequence $\mathcal{S} = \langle U_\alpha \rangle_{\alpha\in\kappa}$ of non-empty open subsets of X there is a subspace $A \subseteq X$ with property Q which meets κ-many terms of the sequence $\mathcal{S}$.*

Furthermore, if Q is a property which is preserved under continuous images and products with compact spaces, then the above conditions are equivalent to:

(3) *The space $X \times K$ is almost cellular-Q for each compact space K.*

Using the same method and the fact that the product of two countably compact spaces, one of which is a k-space, is countably compact (for example see Theorem 3.10.13 in [12]), we have the following two results.

Theorem 4.4 *Suppose that X is a space in which for every family $\mathcal{U}$ of non-empty open sets there is a countably compact subspace of X meeting $|\mathcal{U}|$-many elements of $\mathcal{U}$. Then $X \times L$ is almost cellular-countably-compact whenever L is a countably compact k-space.*

The next result generalizes Corollary 5.15 of [2].

Theorem 4.5 *If X is a cellular-countably-compact space with countable closed-pseudocharacter and Y is a countably compact space, then $X \times Y$ is countably compact.*

Proof It follows from Theorem 2.2 that X is countably compact and first countable, hence a k-space. As in the previous theorem, $X \times Y$ is countably compact. □

It is an immediate consequence of Theorem 5.4 of [2], that if the product of a cellular-compact space X such that $\pi w(X) = \omega$, with $\omega + 1$ is cellular-compact, then the space X is compact. Furthermore, it follows from Theorem 3.13 and Example 3.21 of [21] that if p is a remote point of $\beta\mathbb{R}$, then $\beta\mathbb{R}\backslash\{p\}$ is cellular-compact. It follows that $(\beta\mathbb{R}\backslash\{p\}) \times (\omega+1)$ is not cellular-compact. In a contemporary article [11], whose primary purpose was the construction of a counterexample to Theorem 3.12 of [24], another example is given of a cellular-compact space $\beta\mathbb{Q}\backslash\{p\}$ (where p is a remote point of $\mathbb{Q}$) and a compact space whose product is not cellular-compact (their independently obtained proof is essentially a combination of the results cited in the previous paragraph). The question (first raised in [4]) then arises as to whether the product of a compact space and an almost cellular-compact (respectively, almost cellular-countably-compact) space, is almost cellular-compact (respectively, almost cellular-countably-compact). The following result is a partial answer in the case of almost cellular-compactness.

Theorem 4.6 *Suppose that X is a locally compact, almost cellular-compact T_2-space and K is a compact T_2-space such that $c(X \times K) < \aleph_\omega$, then $X \times K$ is almost cellular-compact.*

Proof By Theorem 5.6 of [4], X is linearly H-closed and hence by Theorem 4.1 of [3], $X \times K$ is also linearly H-closed. Since $c(X \times K) < \aleph_\omega$, and $X \times K$ is locally compact, it is then a consequence of Corollary 4.2 that X is almost cellular-compact. □

Corollary 4.7 *Suppose that p is a remote point of $\mathbb{R}$; the space $Z = (\beta\mathbb{R}\setminus\{p\}) \times (\omega + 1)$ is almost cellular-compact (but not cellular-compact).*

In what follows, $A(\kappa)$ denotes the one-point compactificaction of the discrete space of size κ and $o(X)$ is the cardinality of the topology of X. The proof of the next result is analogous to that of Theorem 3.5 of [1] but for completeness we give a proof.

Theorem 4.8 *For a Hausdorff space X, $X \times K$ is almost cellular-compact (respectively, almost cellular-countably-compact) for each compact Hausdorff space K if and only if $X \times A(\kappa)$ is almost cellular-compact (respectively, almost cellular-countably-compact) for each infinite cardinal $\kappa \leq o(X)$.*

Proof The proofs of the two results are identical and we consider only the case of almost cellular-compactness; furthermore, the necessity is obvious. To prove the sufficiency, suppose that K is a compact space and that $\mathcal{W} = \{U_\alpha \times V_\alpha : \alpha \in \kappa\}$ is a set of mutually disjoint non-empty basic open sets in $X \times K$; then $\langle U_\alpha \rangle_{\alpha \in \kappa}$ is a sequence of open sets in X. We consider the space $X \times A(\kappa)$, and set $\mathcal{W} = \{U_\beta \times \{\beta\} : \beta \in \kappa\}$. Clearly, $\mathcal{W}$ is a cellular family of open sets in $X \times A(\kappa)$ and so there is a compact subspace $C \subseteq X \times A(\kappa)$ which meets κ-many elements of $\mathcal{W}$. It follows that $\pi_X(C]$ is a compact subspace of X which meets κ-many terms of the sequence of sets $\langle U_\alpha \rangle_{\alpha \in \kappa}$ in X. But then $\pi_X[C] \times K$ is a compact subspace of $X \times K$ which meets κ-many elements of $\mathcal{W}$.

Now suppose that $o(X) = \kappa$ and that there is some compact space K such that $X \times K$ is not almost cellular-compact. Then, again by Theorem 4.3, there is some family $\mathcal{U} = \{U_\alpha : \alpha \in \lambda\}$ of non-empty open sets in X with the property that no compact subspace meets λ-many elements of $\mathcal{U}$ and clearly, $\lambda = |\mathcal{U}| \leq o(X)$. Consider the space $X \times A(\lambda)$ and the cellular family of open sets $\mathcal{V} = \{U_\alpha \times \{\alpha\} : \alpha \in \lambda\}$. If there were to exist a compact subspace $T \subseteq X \times A(\lambda)$ which meets λ-many elements of $\mathcal{V}$, then the compact space $\pi_X[T]$ would meet λ-many elements of $\mathcal{U}$, which would be a contradiction. □

The final result of the section is then an immediate consequence of Theorems 4.6 and 4.8.

Corollary 4.9 *Suppose that X is a locally compact, almost cellular-compact T_2-space such that $o(X) < \aleph_\omega$ and K is a compact T_2-space, then $X \times K$ is almost cellular-compact.*

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